Exercise 26

Let $u(x,t)$ be the solution given by (2). Show $u\left(x, t+\frac{3L}{2c}\right)$ is the solution given by (7). What does this imply about the motion described by (2) versus the one described by (7)?

Solution

Equation (2) is a solution to the wave equation on a finite interval with fixed ends and an initial displacement and zero velocity.

$$u(x,t) = \sin\frac{\pi x}{L}\cos\frac{\pi c t}{L} \tag{2}$$

Replace t with $t + \frac{3L}{2c}$.

$$\begin{aligned} u\left(x, t+\frac{3L}{2c}\right) &= \sin\frac{\pi x}{L}\cos\left[\frac{\pi c}{L}\left(t+\frac{3L}{2c}\right)\right] \\ &= \sin\frac{\pi x}{L}\cos\left(\frac{\pi c t}{L}+\frac{3\pi}{2}\right) \\ &= \sin\frac{\pi x}{L}\left(\cos\frac{\pi c t}{L}\underbrace{\cos\frac{3\pi}{2}}_{=\,0} - \sin\frac{\pi c t}{L}\underbrace{\sin\frac{3\pi}{2}}_{=\,-1}\right) \\ &= \sin\frac{\pi x}{L}\sin\frac{\pi c t}{L} \qquad (7) \end{aligned}$$

This is equation (7), which is also a solution to the wave equation on a finite interval with fixed ends but zero displacement and initial velocity. Therefore, whether the string starts with an initial velocity and zero displacement or with zero velocity and an initial displacement, the motion of the string is the same. A string whose motion is described by equation (7) will just take $3L/2c$ units of time to catch up to one whose motion is described by equation (2).